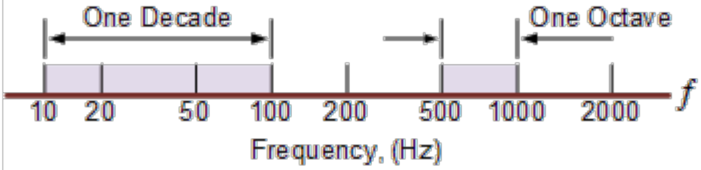


Butterworth Filter Design

In the previous filter tutorials we looked at simple first-order type low and high pass filters that contain only one single resistor and a single reactive component (a capacitor) within their RC filter circuit design.

The Butterworth filter is an analogue filter design which produces the best output response with no ripple in the pass band or the stop band resulting in a maximally flat filter response but at the expense of a relatively wide transition band.

In applications that use filters to shape the frequency spectrum of a signal such as in communications or control systems, the shape or width of the roll-off also called the "transition band". For simple first-order filters this transition band maybe too long or too wide, so active filters designed with more than one "order" are required. These types of filters are commonly known as "High-order" or "n^{th}-order" filters.

The complexity or filter type is defined by the filters "order", and which is dependant upon the number of reactive components such as capacitors or inductors within its design. We also know that the rate of roll-off and therefore the width of the transition band, depends upon the order number of the filter and that for a simple first-order filter it has a standard roll-off rate of 20dB/decade or 6dB/octave.

Then, for a filter that has an n^{th} number order, it will have a subsequent roll-off rate of 20n dB/decade or 6n dB/octave. So a first-order filter has a roll-off rate of 20dB/decade (6dB/octave), a second-order filter has a roll-off rate of 40dB/decade (12dB/octave), and a fourth-order filter has a roll-off rate of 80dB/decade (24dB/octave), etc, etc.

High-order filters, such as third, fourth, and fifth-order are usually formed by cascading together single first-order and second-order filters.

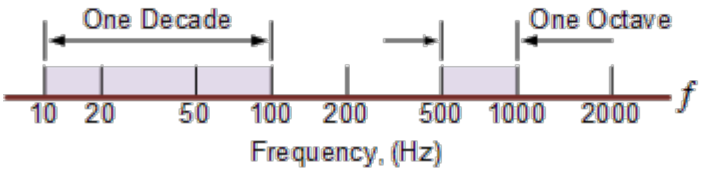
For example, two second-order low pass filters can be cascaded together to produce a fourth-order low pass filter, and so on. Although there is no limit to the order of the filter that can be formed, as the order increases so does its size and cost, also its accuracy declines.

Decades and Octaves

One final comment about *Decades* and *Octaves*. On the frequency scale, a **Decade** is a tenfold increase (multiply by 10) or tenfold decrease (divide by 10). For example, 2 to 20Hz represents one decade, whereas 50 to 5000Hz represents two decades (50 to 500Hz and then 500 to 5000Hz).

An **Octave** is a doubling (multiply by 2) or halving (divide by 2) of the frequency scale. For example, 10 to 20Hz represents one octave, while 2 to 16Hz is three octaves (2 to 4, 4 to 8 and finally 8 to 16Hz) doubling the frequency each time. Either way, *Logarithmic* scales are u extensively in the frequency domain to denote a frequency value when working with am and filters so it is important to understand them.

Logarithmic Frequency Scale



Since the frequency determining resistors are all equal, and as are the frequency determining capacitors, the cut-off or corner frequency (f_C) for either a first, second, third or even a fourth-order filter must also be equal and is found by using our now old familiar equation:

$$fc = \frac{1}{2\pi RC}\ Hz$$

As with the first and second-order filters, the third and fourth-order high pass filters are formed by simply interchanging the positions of the frequency determining components (resistors and capacitors) in the equivalent low pass filter. High-order filters can be designed by following the procedures we saw previously in the Low Pass filter and High Pass filter tutorials. However, the overall gain of high-order filters is **fixed** because all the frequency determining components are equal.

Filter Approximations

So far we have looked at a low and high pass first-order filter circuits, their resultant frequency and phase responses. An ideal filter would give us specifications of maximum pass band gain and flatness, minimum stop band attenuation and also a very steep pass band to stop band roll-off (the transition band) and it is therefore apparent that a large number of network responses would satisfy these requirements.

Not surprisingly then that there are a number of “approximation functions” in linear analogue filter design that use a mathematical approach to best approximate the transfer function we require for the filters design.

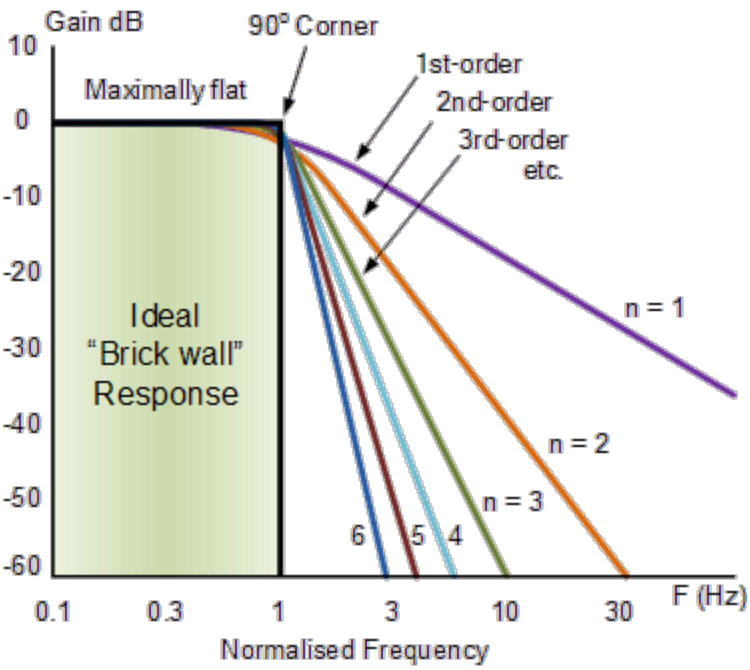
Such designs are known as **Elliptical**, **Butterworth**, **Chebyshev**, **Bessel**, **Cauer** as well as many others. Of these five “classic” linear analogue filter approximation functions only the **Butterworth Filter** and especially the *low pass Butterworth filter* design will be considered here as its the most commonly used function.

Low Pass Butterworth Filter Design

The frequency response of the **Butterworth Filter** approximation function is also often referred to as “maximally flat” (no ripples) response because the pass band is designed to have a frequency response which is as flat as mathematically possible from 0Hz (DC) until the cut-off frequency at -3dB with no ripples. Higher frequencies beyond the cut-off point rolls-off down to zero in the stop band at 20dB/decade or 6dB/octave. This is because it has a “quality factor”, “Q” of just 0.707.

However, one main disadvantage of the Butterworth filter is that it achieves this pass band flatness at the expense of a wide transition band as the filter changes from the pass band to the stop band. It also has poor phase characteristics as well. The ideal frequency response, referred to as a “brick wall” filter, and the standard Butterworth approximations, for different filter orders are given below.

Ideal Frequency Response for a Butterworth Filter



Note that the higher the Butterworth filter order, the higher the number of cascaded stages there are within the filter design, and the closer the filter becomes to the ideal “brick wall” response.

In practice however, Butterworth’s ideal frequency response is unattainable as it produces excessive passband ripple.

Where the generalised equation representing a “nth” Order Butterworth filter, the frequency response is given as:

$$H_{(j\omega)} = \frac{1}{\sqrt{1+\varepsilon^2\left(\frac{\omega}{\omega_p}\right)^{2n}}}$$

Where: n represents the filter order, Omega ω is equal to 2πf and Epsilon ε is the maximum pass band gain, (A_{max}). If A_{max} is defined at a frequency equal to the cut-off -3dB corner point (ƒc), ε will then be equal to one and therefore ε^2 will also be one. However, if you now wish to define A_{max} at a different voltage gain value, for example 1dB, or 1.1220 (1dB = 20*logA_{max}) then the new value of epsilon, ε is found by:

$$H_1 = \frac{H_0}{\sqrt{1+\varepsilon^2}}$$

- Where:
- H_0 = the Maximum Pass band Gain, A_{max}.
- H_1 = the Minimum Pass band Gain.

Transpose the equation to give:

$$\frac{H_0}{H_1} = 1.1220 = \sqrt{1+\varepsilon^2} \quad gives \quad \varepsilon = 0.5088$$

The **Frequency Response** of a filter can be defined mathematically by its **Transfer Function** with the standard Voltage Transfer Function H(jω) written as:

$$H(j\omega) = \left[\frac{Vout(j\omega)}{Vin(j\omega)}\right]$$

- Where:
- Vout = the output signal voltage.
- Vin = the input signal voltage.
- j = to the square root of -1 (√-1)
- ω = the radian frequency (2πf)

Note: (jω) can also be written as (s) to denote the **S-domain.** and the resultant transfer function for a second-order low pass filter is given as:

$$H(s) = \frac{Vout}{Vin} = \frac{1}{S^2 + S + 1}$$

Normalised Low Pass Butterworth Filter Polynomials

To help in the design of his low pass filters, Butterworth produced standard tables of normalised second-order low pass polynomials given the values of coefficient that correspond to a cut-off corner frequency of 1 radian/sec.

n	Normalised Denominator Polynomials in Factored Form
1	$(1+s)$
2	$(1+1.414s+s^2)$
3	$(1+s)(1+s+s^2)$
4	$(1+0.765s+s^2)(1+1.848s+s^2)$
5	$(1+s)(1+0.618s+s^2)(1+1.618s+s^2)$
6	$(1+0.518s+s^2)(1+1.414s+s^2)(1+1.932s+s^2)$
7	$(1+s)(1+0.445s+s^2)(1+1.247s+s^2)(1+1.802s+s^2)$
8	$(1+0.390s+s^2)(1+1.111s+s^2)(1+1.663s+s^2)(1+1.962s+s^2)$
9	$(1+s)(1+0.347s+s^2)(1+s+s^2)(1+1.532s+s^2)(1+1.879s+s^2)$
10	$(1+0.313s+s^2)(1+0.908s+s^2)(1+1.414s+s^2)(1+1.782s+s^2)(1+1.975s+s^2)$

Filter Design – Butterworth Low Pass

Find the order of an active low pass Butterworth filter whose specifications are given as: A_{max}

= 0.5dB at a pass band frequency (ωp) of 200 radian/sec (31.8Hz), and A_{min} = -20dB at a stop band frequency (ωs) of 800 radian/sec. Also design a suitable Butterworth filter circuit to match these requirements.

Firstly, the maximum pass band gain A_{max} = 0.5dB which is equal to a gain of **1.0593**, remember that: 0.5dB = 20*log(A) at a frequency (ωp) of 200 rads/s, so the value of epsilon ε is found by:

$$1.0593 = \sqrt{1 + \varepsilon^2}$$

$$\therefore \varepsilon = 0.3495 \quad and \quad \varepsilon^2 = 0.1221$$

Secondly, the minimum stop band gain A_{min} = -20dB which is equal to a gain of **10** (-20dB = 20*log(A)) at a stop band frequency (ωs) of 800 rads/s or 127.3Hz.

Substituting the values into the general equation for a Butterworth filters frequency response gives us the following:

$$H(j\omega) = \frac{H_0}{\sqrt{1 + \varepsilon^2 \left(\frac{\omega_S}{\omega_P}\right)^{2n}}}$$

$$\frac{1}{10} = \frac{1}{\sqrt{1 + 0.1221 \left(\frac{800}{200}\right)^{2n}}}$$

$$(10)^2 = 1 + 0.1221 \; x \; 4^{2n}$$

$$\therefore 100 - 1 = 0.1221 \; x \; 4^{2n}$$

$$4^{2n} = \frac{99}{0.1221} = 810.811$$

$$4^n = \sqrt{810.811} = 28.475$$

$$\therefore n = \frac{\log 28.475}{\log 4} = 2.42$$

Since n must always be an integer (whole number) then the next highest value to 2.42 is n = 3, therefore a **"a third-order filter is required"** and to produce a third-order **Butterworth filter**, a second-order filter stage cascaded together with a first-order filter stage is required.

From the normalised low pass Butterworth Polynomials table above, the coefficient for a third-order filter is given as $(1+s)(1+s+s^2)$ and this gives us a gain of 3-A = 1, or A = 2. As A = 1 + (Rf/R1), choosing a value for both the feedback resistor Rf and resistor R1 gives us values of 1kΩ and 1kΩ respectively as: (1kΩ/1kΩ) + 1 = 2.

We know that the cut-off corner frequency, the -3dB point (ω_o) can be found using the formula 1/CR, but we need to find ω_o from the pass band frequency ω_p then,

$$H(j\omega) = \frac{H_0}{\sqrt{1+\varepsilon^2\left(\frac{\omega_o}{\omega_p}\right)^{2n}}}$$

$$3dB = 1.414 \ at \ \omega = \omega_o$$

$$\frac{1}{1.414} = \frac{1}{\sqrt{1+\varepsilon^2\left(\frac{\omega_o}{\omega_p}\right)^{2n}}}$$

$$2 = 1 + \varepsilon^2\left(\frac{\omega_o}{\omega_p}\right)^{2n}$$

$$\therefore 1 = \varepsilon\left(\frac{\omega_o}{\omega_p}\right)^{n}$$

$$\omega_o{}^n = \frac{\omega_p{}^n}{\varepsilon}$$

$$\omega_o{}^3 = \frac{200^3}{0.3495}$$

$$\omega_o{}^3 = 22.889 \ x \ 10^6$$

$$\therefore \omega_o = 283.93 \simeq 284 rads/s$$

So, the cut-off corner frequency is given as 284 rads/s or 45.2Hz, (284/2π) and using the familiar formula 1/CR we can find the values of the resistors and capacitors for our third-order circuit.

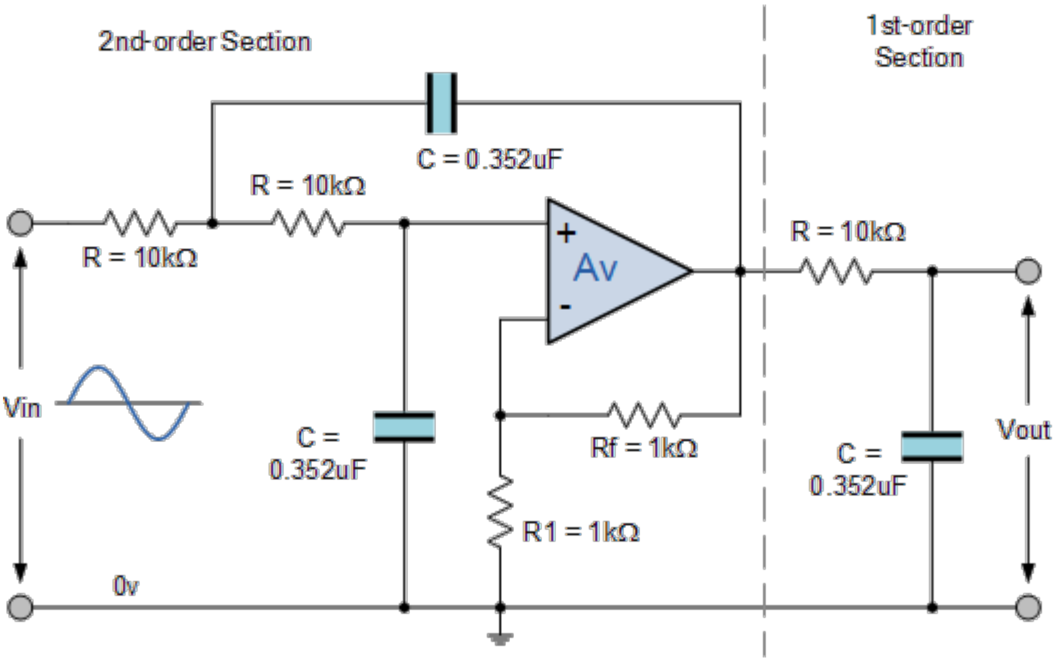
$$284 rads/s = \frac{1}{CR} \quad use \ a \ value \ of \ R = 10k\Omega$$

$$\therefore Capacitor \ C = \frac{1}{284 \ x 10{,}000} = 0.352 uF$$

Note that the nearest preferred value to 0.352uF would be 0.36uF, or 360nF.

Third-order Butterworth Low Pass Filter

and finally our circuit of the third-order low pass **Butterworth Filter** with a cut-off corner frequency of 284 rads/s or 45.2Hz, a maximum pass band gain of 0.5dB and a minimum stop band gain of 20dB is constructed as follows.



So for our 3rd-order Butterworth Low Pass Filter with a corner frequency of 45.2Hz, C = 360nF and R = 10kΩ

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- 2. Passive Low Pass Filter
- 3. Passive High Pass Filter
- 4. Passive Band Pass Filter
- 5. Active Low Pass Filter
- 6. Active High Pass Filter
- 7. Active Band Pass Filter
- 8. Butterworth Filter Design
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- 11. Band Stop Filter
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75 Comments

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- *Rodney Etheridge*

  Very nice tutorial session. I simulated the results and obtained excellent results. Thanks for the information.

  Rodney

  Posted on May 18th 2024 | 3:14 pm
  Reply
- *Josh*

  I was wondering what to do if the order N is equal to zero? like Amax = 40dB Amin = -40dB ωp = 100 ωs = 400

  Posted on April 22nd 2024 | 1:36 pm
  Reply
- *Obasi Richard Ubadire*

  Am ready to learn

  Posted on January 10th 2024 | 10:28 am
  Reply
- *Douglas Goldfarb*

  I am interested in the three dimensional fast Fourier on different brains from beglitiers nuerodynamics laboratory suny downstate and through mit cognitive science the full Neural network of people and statistical analysis

  Posted on November 25th 2022 | 1:05 am
  Reply
- *John May*

  I can't make sense of that circuit. I'm not saying it doesn't work. But the signal going through that top capacitor a) bypasses the opamp, b) is mostly pulled to ground, and c) participates in the negative feedback side of the op-amp. I can see TWO stages here, and somehow that other signal path must count for a third stage… but I just can't figure out how. Kind of wish there were an explanation. I mean… it's not cascaded. It's… something else.

  Posted on August 03rd 2022 | 6:21 pm
  Reply
- *Sean*

  Thank you for the information. I enjoyed this refresher course.

  Posted on May 16th 2022 | 3:07 am
  Reply
- *jaya dwivedi*

goo explanation

Posted on October 08th 2020 | 6:09 pm
Reply

- *nikhil dwivedi*

  it's goo or good? :ps

  Posted on November 06th 2020 | 7:09 am
  Reply

- *LALIT TUTY*

  (A+B+C(G)=22/7

  Posted on September 08th 2020 | 1:52 pm
  Reply

- *Levi Mays*

  Also, though the denominator table was mentioned in the worked-out example, it seemed to have no relevance on the circuit design. This needs attention.

  Posted on March 26th 2020 | 3:41 pm
  Reply

- *Levi Mays*

  Needed mentions:

  Prior to this page, one should read Sallen-Key design Filters then Second-Order Filters. In example, the filter's overall passband gain is used to find 'n'. For second-order Sallen-Key filter gain, let Q be 0.5. This won't effect overall passband gain since the resistors in the successive filter stages will compensate of S-K gain.

  Posted on March 26th 2020 | 3:40 pm
  Reply

- *Anas Muhammad Buhari*

  Interesting Butterworth filter design

  Posted on March 02nd 2020 | 9:41 pm
  Reply

- *FARZAD*

  I have designed a dual pass filter with layout Ads now i can't get LC equivalent circuit please guide me?

  Posted on February 02nd 2020 | 5:39 pm
  Reply

- *Jhake Rivera*

  Good!

  Posted on January 21st 2020 | 8:49 am
  Reply

- *sai Teja Tutika*

  gioooooooood

  Posted on September 04th 2019 | 10:35 am
  Reply

- *Ester*

How about Butterworth unnormalized for highpass and bandpass filter

Posted on [May 07th 2019 | 4:40 pm]
[Reply]

- *Elvis*

  In a fourth **order filter** the gain of the first section is 1.152 while that of the second section is 2.235. These gain values are necessary to guarantee **Butterworth** resporise and must remain the same regardless of the **filter** cutoff frequency.

  Posted on [April 03rd 2019 | 9:12 am]
  [Reply]

- *Ismail Omary*

  Amazing!!!

  Posted on [February 10th 2019 | 2:02 pm]
  [Reply]

- *Kenny*

  Hi Wayne, there is a section that says "Where: n represents the filter order, Omega ω is equal to 2πƒ and Epsilon ε is the maximum pass band gain, (Amax)", where there appear to be sections of text missing. For example, Epsilon is not actually defined. And Amax is supposed to be the maximum allowed passband linear attenuation, rather than maximum passband linear gain, right? There is also at description that says "H0 = the Maximum Pass band Gain, Amax.", but the "Amax" word should be omitted right? Also, to avoid confusion between gain in dB and linear gain, then H0 should be called 'max passband linear gain' maybe, rather than just 'max passband gain'. Also, are those plots normalised to the cutoff frequency, wo? Not normalised to the passband edge frequency wp, right? Thanks Wayne.

  Posted on [January 08th 2019 | 7:32 am]
  [Reply]

  - *Wayne Storr*

    Epsilon (e) is the passbands maximum tolerance or gain for the ripple where the ripple is defined as the peak to peak variation in dB within the passband. Since the maximum ripple, Amax is one, that is 0 dB, the minimum ripple is given as Amin = 1/(1 + e^2)

    H0 is commonly defined as the maximum gain (Amax) at DC and H1 as the passband edge (Amin) before roll-off, thus the passband ripple varies between H0 and H1

    Correct, the example is normalised to the cutoff frequency, wo at H0-3dB

    Posted on [January 08th 2019 | 5:58 pm]
    [Reply]

- *Kenny*

  Hello there. The formulae that says H(jw) = Ho/sqrt[etc] …. the left-hand side should be magnitude |H(jw)|, right? Not H(jw). The left hand-side should really be square root of H(jw).H*(jw), where H*(jw) is the complex conjugate of H(jw). Also, how does the epsilon value relate to butterworth magnitude functions that don't contain 'epsilon'? It appears there are other sources that focus on magnitude functions where the epsilon value is 1. Thanks!

  Posted on [November 10th 2018 | 12:13 am]
  [Reply]

- *Kaleb*

0.352 uF huh? I think your units are incorrect. The answer to the capacitor is 0.352mF not uF, micro Farads would be 352uF.

Posted on November 06th 2018 | 5:36 pm
Reply

- More